Uniform volume doubling for the unitary group $U(2)$

Nathaniel Eldredge
University of Northern Colorado
Joint work with:
Maria Gordina (UConn)
Laurent Saloff-Coste (Cornell)

Stochastics and Geometry
Banff International Research Station
12 September 2024

Outline

- What is uniform doubling?
- Why do we care about it?
- How have we proved it for $\mathrm{SU}(2)$?
- How do we prove it for $\mathrm{U}(2)$ and relatives?
- What's next?

Setup

- G is a connected unimodular Lie group with identity e
- g denotes a left-invariant Riemannian metric on G (allowed to vary)
- Note that g can also be viewed as an inner product on the Lie algebra $\mathfrak{g}$
- Riemannian distance, volume, balls, gradient, Laplacian, etc, all depend on the choice of g
- $\mathfrak{L}(G)$ denotes the set of all left-invariant Riemannian metrics on G

Volume doubling

We say (G, g) is volume doubling with constant D_g if we have

$$\mathrm{Vol}_g(B_g(e, 2r)) \leq D_g \mathrm{Vol}_g(B_g(e, r))$$

for all $r > 0$.

- By left-invariance, the center point of the ball is irrelevant
- We could replace the Riemannian volume Vol_g by any fixed normalization μ of the bi-invariant Haar measure, as they differ only by a constant multiple
- Doubling is always an interesting property, but is particularly interesting on Lie groups...

Doubling and Poincaré

Theorem (Varopoulos 1987; Kleiner 2010; Saloff-Coste 2002)

Suppose G is a connected unimodular Lie group, g is a left-invariant Riemannian metric, and (G, g) is volume doubling with constant D_g. Then we have the scale-invariant Poincaré inequality

$$\int_{B(r)} |f - f_r|^2 \, d\,\mathrm{Vol} \leq 2D_g r^2 \int_{B(2r)} |\nabla f|^2 \, d\,\mathrm{Vol}$$

where f_r is the average of f over $B(r)$, and $B(r)$, Vol, ∇ are all with respect to g.

Note the constant is not merely existential, but is actually $2D_g$.

Further consequences of doubling on Lie groups

Doubling plus Poincaré is well known to imply a “constellation” of other functional inequalities:

- Parabolic Harnack inequalities

$$\sup_{Q_-} u \le C_g \inf_{Q^+} u \qquad (\partial_t - \Delta)u = 0, \quad u > 0$$

[Grigor’yan 1991; Saloff-Coste 1992]

- Heat kernel upper, lower, and time derivative bounds, e.g.

$$p_t(x,y) \le C_g \frac{\left(1 + \frac{d(x,y)^2}{4t}\right)^{\kappa_g}}{V(\sqrt{t})} \exp\left(-\frac{d(x,y)^2}{4t}\right)$$

[Grigor’yan 1994; Sturm 1995, 1996; Carron 1996; Saloff-Coste 1992, 2002; Coulhon–Sikora 2008]

Yet more consequences of doubling plus Poincaré

When G is compact, doubling plus Poincaré also implies:

- Spectral gap estimates and eigenvalue asymptotics, e.g.

$$\lambda_1 \le \frac{C_g}{\operatorname{diam}(G)^2}$$

  [EGS; Maheux–Saloff-Coste 1995; cf. P. Li 1980, Judge–Lyons 2017]

- (G compact) Ergodicity or "mixing time" estimates for Brownian motion, e.g.

$$\frac{c_g}{V(\sqrt{t})} e^{-2\lambda_1 t} \le \|p_t - \mathbf{V}^{-1}\|_{L^2}^2 \le \frac{C_g}{V(\sqrt{t})} e^{-2\lambda_1 t}$$

Dependence of the constants

- By careful inspection of the proofs, one can verify that the foregoing inequalities all hold with the constants C_g depending on g only through its doubling and Poincaré constants.
- In the Lie group setting, those are the same constant! So all of these hold with constants depending **only** on the doubling constant D_g.

Uniform doubling

- For a fixed group G, suppose it could be shown that there was a uniform upper bound for D_g over all left-invariant Riemannian metrics g:

$$D(G) := \sup_{g \in \mathfrak{L}(G)} D_g < \infty$$

- We would then say that G is **uniformly doubling**.
- Then the "constellation" of inequalities would hold with constants *independent* of the metric.
- This would also cover the left-invariant sub-Riemannian metrics!

Which groups are uniformly doubling?

- This trivially holds for abelian Lie groups: $\mathbb{R}^n$, tori $(S^1)^n$, etc. Indeed $D(\mathbb{R}^n) = D((S^1)^n) = 2^n$.
- What about non-trivial examples?
- Ricci curvature lower bounds imply doubling (Bishop–Gromov comparison theorem). But this will not help, as there is typically no uniform lower Ricci bound over all left-invariant metrics $g \in \mathfrak{L}(G)$.

Examples and a conjecture

Theorem (E.–Gordina–Saloff-Coste 2018)

The 3-dimensional special unitary group $\mathrm{SU}(2)$ *is uniformly doubling.*

Theorem (EGS 2024+)

The 4-dimensional unitary group $\mathrm{U}(2)$ *is uniformly doubling, as is every Lie group whose universal cover is* $\mathrm{SU}(2) \times \mathbb{R}^n$ *for any* n*, with the constant depending only on* n*.*

Conjecture

Every compact connected Lie group is uniformly doubling.

Quotients

Theorem (Folklore)

If G is uniformly doubling, and $N \trianglelefteq G$ is a closed normal subgroup, then the quotient $H = G/N$ is uniformly doubling, with $D(H) \leq D(G)^2$.

- Each left-invariant Riemannian metric h on H can be lifted to some g on G, such that the quotient map $\pi : (G, g) \to (H, h)$ is a Riemannian submersion. In particular it maps balls to balls.
- Use a clever but elementary Fubini argument [Guivarc'h 1973] on a well-chosen set $P \subset G \times H$ to show

$$\mu_G(B_g(r))\mu_H(B_h(2r)) \leq (\mu_G \times \mu_H)(P) \leq \mu_G(B_g(3r))\mu_H(B_h(r))$$

which rearranges to

$$\frac{\mu_H(B_h(2r))}{\mu_H(B_h(r))} \leq \frac{\mu_G(B_g(3r))}{\mu_G(B_g(r))}$$

and implies $D_h \leq D_g^2$.

Products?

- Since uniform doubling passes to quotients, then for our theorem, it suffices to study $\mathrm{SU}(2) \times \mathbb{R}^n$.
- We showed previously that $\mathrm{SU}(2)$ is uniformly doubling, and of course $\mathbb{R}^n$ is, so why is this not trivial?
- What's trivial is that if G, G' are each uniformly doubling, then so is every left-invariant *product* metric on $G \times G'$; i.e. those for which $TG \perp TG'$.
- But we must consider *every* left-invariant metric, including those for which the angle between TG, TG' is arbitrarily close to 0.
- Reasonable to conjecture that a product of uniformly doubling groups is uniformly doubling in general, but we have no proof.

Milnor bases

- We prove uniform doubling for $\mathrm{SU}(2)$ by considering a nice parametrization of $\mathfrak{L}(\mathrm{SU}(2))$, then computing sharp upper and lower estimates for the volume $\mu(B_g(r))$ in terms of these parameters.
- The parametrization is based on the idea of a **Milnor basis**.

Theorem (Milnor 1976)

For every left-invariant Riemannian metric on $\mathrm{SU}(2)$, viewed as an inner product on the Lie algebra $\mathfrak{su}(2)$, there is a basis $u_1, u_2, u_3 \in \mathfrak{su}(2)$ which is orthogonal for g, and which satisfies the Lie bracket relations

$$[u_1, u_2] = u_3, \quad [u_2, u_3] = u_1, \quad [u_3, u_1] = u_2.$$

Moreover, the metric g is uniquely determined up to isometric isomorphism by the three values $a_i^2 = g(u_i, u_i)$.

We can think of a_i as the "cost" of moving in the u_i direction.

Volume lower bounds in SU(2), streamlined version

- Let g have parameters (a_1, a_2, a_3). To estimate $\mu(B_g(r))$ from below, we try to identify a large enough set of group elements contained in it.
  (We want to avoid having to explicitly compute geodesics.)
- Question: for which values of x_1 do we have $e^{x_1 u_1} \in B_g(r)$?
- Clearly this holds for $|x_1| \le r/a_1$, since the path $t \mapsto e^{tu_1}$ has speed a_1.
- But we can also produce e^{xu_1} via the approximate commutation relation

  $$e^{stu_1} \approx e^{su_2} e^{tu_3} e^{-su_2} e^{-tu_3}$$

  which comes from the leading terms of the Baker–Campbell–Hausdorff–Dynkin formula $e^A e^B = \exp(A + B + \frac{1}{2}[A, B] + \dots)$.
- That would be the right way to go in sub-Riemannian geometry, if u_1 were not a horizontal direction (corresponding to $a_1 = +\infty$).

Lower bounds continued

$$e^{stu_1} \approx e^{su_2} e^{tu_3} e^{-su_2} e^{-tu_3}$$

- Ignoring the $\approx$, this will be in $B_g(r)$ provided that $e^{su_2}, e^{tu_3} \in B_g(r/4)$ (recall distance is left invariant), which happens when $|s| \le r/4a_2$, $|t| \le r/4a_3$.
  So we also get $e^{x_1 u_1}$ for $|x_1| \le r^2/(16a_2a_3)$.
- In summary,
  $$e^{x_1 u_1} \in B_g(r) \text{ for } |x_1| \le \max\left(\frac{r}{a_1}, \frac{r^2}{16a_2a_3}\right)$$
  and similarly for $e^{x_2u_2}, e^{x_3u_3}$ by permuting indices.
- Handling the remainder can be avoided via a somewhat more complicated formula which writes e^{xu_1} exactly as a product of seven elements of the form e^{su_2}, e^{tu_3}.
  (Again, relies on the bracket relations $[u_i, u_j] = u_k$.)

Finishing lower bounds

- Define "coordinates of the second kind" $\Psi : \mathbb{R}^3 \to \mathrm{SU}(2)$ as

$$\Psi(x_1, x_2, x_3) = e^{x_1 u_1} e^{x_2 u_2} e^{x_3 u_3}$$

- Ignoring constants, we've shown that $\Psi(K) \subset B_g(3r)$ where K is the box defined by $|x_i| \le \max\left(\frac{r}{a_i}, \frac{r^2}{a_j a_k}\right)$.
- If we further require $|x_i| \le 1$, then Ψ is injective on K and we can bound its Jacobian below. So we get

$$\mu(B_g(3r)) \ge c|K| \ge \prod_{i=1}^{3} \min\left(\max\left(\frac{r}{a_i}, \frac{r^2}{a_j a_k}\right), 1\right)$$

Upper bounds

- Here we try to put a ball inside a box: given an element $x \in B_g(r)$, show that x can be written as
$$x = e^{x_1 u_1} e^{x_2 u_2} e^{x_3 u_3}$$
with suitable bounds on x_1, x_2, x_3, thus showing $B_g(r) \subset \Psi(K')$ for some other box $K' \subset \mathbb{R}^3$.
- We can easily bound the Jacobian of Ψ from above, so this gives us a bound in terms of the volume of K.
- Fix a path $\gamma : [0,1] \to \mathrm{SU}(2)$ from e to x with speed $|\dot{\gamma}(t)| < r$. We can write γ in the Ψ coordinates as
$$\gamma(t) = \Psi(x_1(t), x_2(t), x_3(t)) = e^{x_1(t) u_1} e^{x_2(t) u_2} e^{x_3(t) u_3}$$

Upper bounds continued

- Look at the Maurer–Cartan form $\sigma(t) = (L_{\gamma(t)}^{-1})_* \dot{\gamma}(t) \in \mathrm{su}(2)$, i.e. the velocity of γ pulled back to the identity, viewed as a path in the Lie algebra. We can write

$$\sigma(t) = \alpha_1(t)u_1 + \alpha_2(t)u_2 + \alpha_3(t)u_3$$

  with $|\alpha_i(t)| \leq r/a_i$ because of the speed constraint.
- Compute the Maurer–Cartan form of $\Psi(x_1(t), x_2(t), x_3(t))$ using some exponential identities (based again on $[u_i, u_j] = u_k$).
- Setting equal to $\sigma(t)$, and equating coefficients of u_i, gives an explicit system of ODEs for the $x_i(t)$ driven by the $\alpha_i(t)$.
- Then some simple *a priori* estimates provide bounds on $|x_i(t)|$ and in particular $|x_i(1)|$, which match the lower bounds.

Extending to $G = \mathrm{SU}(2) \times \mathbb{R}^n$

- We again want, for each g, to find a "nice" basis for the Lie algebra $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbb{R}^n$, with simple inner product and Lie algebra relations
- We can still find a Milnor basis u_1, u_2, u_3 for the $\mathfrak{su}(2)$ summand, but its g-orthogonal complement need not be in the center $\mathbb{R}^n$, giving more nonzero Lie brackets
- Instead, we construct vectors $v_i = u_i + d \cdot f_i$ such that:
  - the vectors v_i are orthogonal to each other, with some norms $|v_i|_g = a_i$
  - the v_i are orthogonal to the center of $\mathfrak{g}$;
  - f_i are in the center, and d is some scalar.
- In particular, $[v_i, v_j] = u_k$.
- Moreover, via a lifting argument, we reduce to the case of $\mathrm{SU}(2) \times \mathbb{R}^3$, where the f_i are orthonormal.

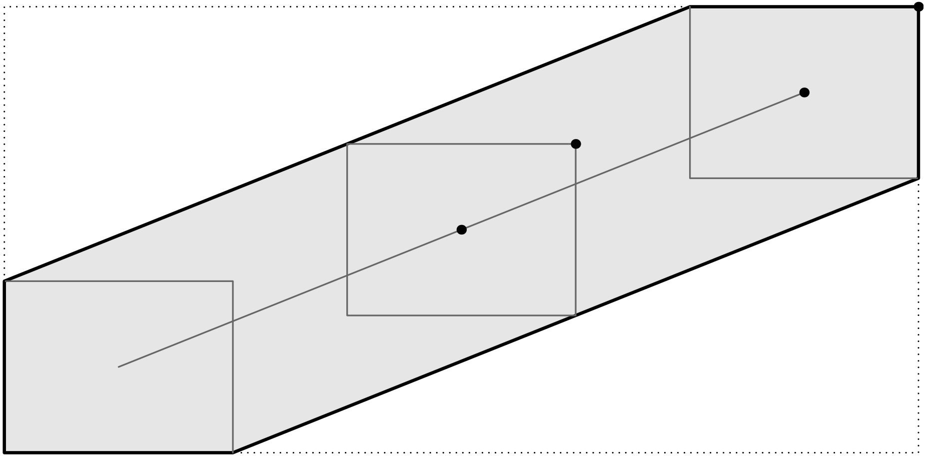
Lower bounds

So, which elements of the form $e^{x_1 v_1 + y_1 f_1}$ are in $B_g(r)$?

- $e^{x_1 v_1}$ for $|x_1| \leq r/a_1$
- $e^{y_1 f_1}$ for $|y_1| \leq r$
- $e^{t u_1} = e^{t(v_1 - d f_1)}$ for $t \leq C \frac{r^2}{a_2 a_3}$, via the bracket $[v_2, v_3] = u_1$
- Likewise for $t \leq C \frac{r^3}{a_1 a_2^2}$, via the third-order bracket $[v_2, [v_1, v_2]] = [v_2, u_3] = u_1$
- Likewise for $t \leq C \frac{r^3}{a_1 a_3^2}$, via the bracket $[[v_3, v_1], v_3] = [u_2, v_3] = u_1$

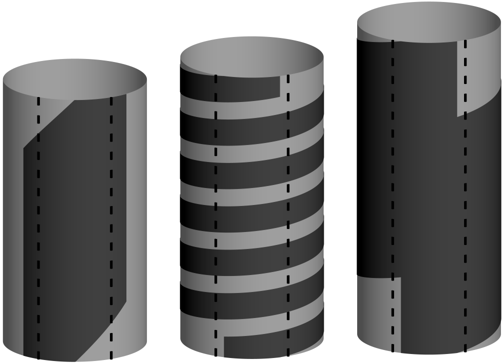
A hexagon. . .

Thus the relevant set of (x_1, y_1) contains all points that can be reached via a combination of horizontal, vertical, and diagonal lines:



… wrapped around a cylinder

But $e^{x_1 u_1}$ is periodic in u_1 (traces a great circle in $\mathrm{SU}(2)$). So in fact the hexagon is wrapped around a cylinder, possibly overlapping itself:



(a) $d = 1$, $\mu^* = 0.8$, $\nu^* = \pi/4$, $\xi^* = 3$

(b) $d = 0.1$, $\mu^* = 0.8$, $\nu^* = 12\pi$, $\xi^* = 0.3$

(c) $d = 0.2$, $\mu^* = 0.8$, $\nu^* = 4\pi$, $\xi^* = 2$

Finishing the bounds

- We have to compute (at least up to a constant) the surface area of the “wrapped hexagons” from the previous slide.
- Finally, we have that the ball $B_g(r)$ contains the image, under some “coordinates of the second kind” Ψ, of the product of three such wrapped hexagons. This yields the lower bounds.
- For the upper bounds, we use ODEs as before, obtaining *a priori* bounds that likewise constrain (x_i, y_i) inside a wrapped hexagon.

Future directions

- More examples of uniformly doubling groups
- What other Lie algebras admit nice bases? (Representation theory)
- Greater focus on doubling as a substitute for Ricci bounds
- More general and robust techniques

The end

Thanks for listening!